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On the Singularities of Analytic Functions Defined by L-Dirichletian Elements

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In this note, using the Dzrbasjan–Mandelbrojt inequality, we locate the singularities of analytic functions defined by L-Dirichletian elements. Though Dzrbasjan remarks that his inequality can be used to locate the singularities of functions defined by the Dirichlet–Taylor series, it is the convergence theory we established earlier [M. Blambert and M. Berland, *C. R. Acad. Sci. Paris* **280** (1975), 263–266; M. Blambert and R. Parvatham, *Ann. Inst. Fourier* **29** (1979), 239–262] that leads us exactly to determine the natural boundaries of such functions, whereas Dzrbasjan is never concerned with the convergence of such a series, as he himself states.

Let us consider an L-Dirichletian element,

$$\{f\}\colon \sum_{n=1}^{\infty} P_n(s)\exp(-\lambda_n s), \qquad s=\sigma+i\tau,\ (\sigma,\tau)\in\mathbb{R}^2, \tag{1}$$

where $P_n(s)$ is a complex polynomial of degree m_n, $(\lambda_n)_1^\infty$ is a D-sequence (that is, a positive, strictly increasing, unbounded sequence) with upper density $D^*<\infty$. Let $P_n(s)=\sum_{j=0}^{m_n} a_{n,j}s^j$, where $a_{n,m_n}\neq 0$; let $\mathscr{E}_n$ be the set of points of $\mathbb{C}$ which are zeros for P_n and let $\mathscr{E}_\infty$ be the set of points of $\mathbb{C}$ which are zeros for an infinity of polynomials P_n. Let us put $\mathscr{E}=\bigcup_n \mathscr{E}_n$ and $\mathscr{E}^*=\mathscr{E}^d\cup\mathscr{E}_\infty$, where $\mathscr{E}^d$ is the derived set of $\mathscr{E}$, and let us suppose that $\mathbb{C}-\mathscr{E}^*\neq\varnothing$. Let us define

$$\forall_{s\in\mathbb{C}-\mathscr{E}^*}\quad \delta_*(s)=\liminf_{n\to\infty}\left\{\frac{-\mathrm{Log}\,|P_n(s)\exp(-\lambda_n s)|}{\lambda_n}\right\}$$

and

$$\mathscr{D}_{*\alpha}=\{s\in\mathbb{C}-\mathscr{E}^*\mid \delta_*(s)>\alpha\}.$$

We know that [1, 2] under the conditions $D^*<\infty$ (which implies trivially that $\lim_{n\to\infty}\log n/\lambda_n=0$) and $\beta^*=\limsup_{n\to\infty}(m_n/\lambda_n)<\infty$, the series (1)

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converges absolutely in $\mathscr{D}_{*0}$ and diverges in $\mathbb{C} - \mathscr{E}^* - \bar{\mathscr{D}}_{*0}$, where $\bar{\mathscr{D}}_{*0}$ is the closure of $\mathscr{D}_{*0}$. Let us put

$$L(s) = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{\lambda_n^2}\right)^{m_n+1},$$

$$M_n = \operatorname{Max} \left\{ \frac{|\gamma_{n,i}|}{i!} \,\middle|\, i \in \{0, 1,..., m_n\} \right\}$$

(where $\gamma_{n,i}$ is the value at $s = \lambda_n$ of the ith derivative of the function $s \mapsto (s-\lambda_n)^{m_n+1}/L(s)$, which is defined by continuity at the point λ_n),

$$\gamma = \limsup_{n\to\infty} \left\{ \frac{\operatorname{Log} M_n}{\lambda_n} \right\}$$

and

$$\gamma^+ = \operatorname{Max}(0, \gamma).$$

Denoting by $\bar{D}^*$ the mean upper density of the sequence (λ_n) [4, 5], we can prove the following:

THEOREM 1. *Under the conditions*

$$D^* < \infty, \qquad \beta^* < \infty, \qquad |\gamma| < \infty \qquad \text{and} \qquad \mathscr{D}_{*,0} \neq \varnothing,$$

we cannot obtain a holomorphic extension of the function $\mathscr{D}_{*0} \ni s \to f(s)$ (*where* $f(s)$ *is the sum of the series* (1) *at the point* s) *to the open set* $\mathscr{D}_{*,-\gamma^+} \cup (\bigcup_{s \in \operatorname{Fr}\mathscr{D}_{*,-\gamma^+}} d(s, \pi R))$ *with* $R > \bar{D}^*$.

Proof. Following the notations of Dzrbasjan [4], using the Dzrbasjan–Mandelbrojt inequality [5, 4] and proceeding as in another of our papers ([3, see inequality (II.3)]), we get

$$\left| \sum_{j=0}^{m_n} a_{n,j} s_0^j \right| \leqslant \left[\frac{\pi^2 R}{2} \exp(2\nu(R)) \right] \omega_n \lambda_n M^f(s_0, \pi R) \exp[\lambda_n \sigma_0],$$

where ν is the excess function for Dzrbasjan sequence [4], $M^f(s_0, \pi R)$ is the maximum modulus of the holomorphic extension of f to the disc $d(s_0, \pi R)$ along a channel of width $2\pi R$, with the Jordan arc joining the point s_0 to a point s_1 as its axis, such that $d(s_1, \pi R) \subset \mathscr{D}_{*0}$. Hence

$$|P_n(s_0) \exp(-s_0 \lambda_n)| \leqslant A(R)\, \lambda_n \omega_n M^f(s_0, \pi R).$$

Now for $s_0 \in \mathbb{C} - \mathscr{E}^*$, we have

$$\delta_*(s_0) = \liminf_{n\to\infty} \left\{ \frac{-\text{Log}\, |P_n(s_0) \exp(-s_0 \lambda_n)|}{\lambda_n} \right\}$$

$$\geqslant \liminf_{n\to\infty} \left\{ \frac{-\text{Log}\, \omega_n}{\lambda_n} \right\}.$$

We know that [3, see inequality (II.2)]

$$\limsup \left\{ \frac{\text{Log}\, \omega_n}{\lambda_n} \right\} \leqslant \gamma \leqslant \gamma^+.$$

Hence, finally we have

$$\delta_*(s_0) \geqslant -\gamma^+ \ (\Rightarrow s_0 \in \mathscr{D}_{*-\gamma^+}).$$

Now let us suppose that there exists a point $s_0' \in \text{Fr}(\mathscr{D}_{*,-\gamma^+-\varepsilon})$ with $\varepsilon > 0$, such that f admits a holomorphic extension to $d(s_0', \pi R)$ along a channel of width $2\pi R$ with the Jordan arc joining s_0' to s_1 as axis, such that $d(s_1, \pi R) \subset \mathscr{D}_{*,0}$. By definition we have $\delta_*(s_0') = -(\gamma^+ + \varepsilon) < -\gamma^+$. But as a consequence of the result established above, we must have $\delta_*(s_0') \geqslant -\gamma^+$. Hence a contradiction establishes the truth of the theorem.

Now we also have

COROLLARY 2. *If* $\gamma^+ = 0$ *and* $\bar{D}^* = 0$, *then* $\text{Fr}(\mathscr{S})$ *is a natural boundary for the analytic function defined by* (1) *in* $\mathscr{S}$, *where* $\mathscr{S}$ *is a connected component of* $\mathscr{D}_{*0}$. *If* $\mathscr{D}_{*0}$ *has more than one connected component, then to each connected component there corresponds an analytic function defined by* (1) *in that component.*

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